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Relativistic Fluid Spheres and Noncomoving Coordinates. I

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Abstract

Spherically symmetric relativistic spheres of perfect fluid are defined to be isotropic by Walker's (1935) isotropy condition. This condition permits the use of noncomoving coordinate systems, which, it is argued, are preferable to comoving systems in certain situations. It is assumed that these systems are such that the metric is orthogonal and involves three unknown functions. These functions are obtained by solving the equation expressing the isotropy condition in a number of cases defined by ancillary mathematical assumptions. Formulas are given for the pressure, density, and velocity components of the fluid, but the detailed physical analysis of the various cases found is reserved for a subsequent paper.

1. *Introduction*

In recent years the problem of gravitational collapse has stimulated the search for new solutions of Einstein's equations of general relativity such as those produced by Bonnor and Faulkes (1967), McVittie (1967), Whitrow and Thompson (1967), and Cahill and Taub (1971) for spherically symmetric distributions. It is almost always the case that the results are described in terms of comoving coordinate systems. These systems have, of course, long been employed in the cosmology of general relativity as well as in such early investigations as those of McVittie (1933), Bondi (1947), and Kustaanheimo and Quist (1948). They have the advantage of simplifying the mathematical analysis to a great extent, and this property no doubt explains their popularity. However, the principle of covariance implies that noncomoving systems may equally well be used even if, as will be briefly indicated in Section 2, a comoving system can always be established for any particular spherically symmetric situation. But it cannot be guaranteed that a solution of Einstein's equations obtained in terms of elementary functions by the use of noncomoving coordinates will still appear in a simple form when it is transformed to the appropriate comoving system. An example of a similar property is found in the case of the

Schwarzschild space-time. The coefficients of its metric involve the elementary functions $(1 - 2m/r)$, $(1 - 2m/r)^{-1}$, and r^2 in the usual coordinate system, but the coefficients are converted to transcendental functions when Kruskal (1960) coordinates are employed.

Consequently we have thought it worth while to examine the possibility of solving Einstein's equations in terms of noncomoving coordinates. The few previous investigations of this kind are discussed in Section 3. We shall be concerned in the present paper with the determination of solutions by making purely mathematical simplifying assumptions. These do not of themselves guarantee that the structure and properties of the fluid spheres obtained are acceptable from the physical point of view. The physical interpretations of our results will be given in a subsequent paper.

The configurations to be discussed will be assumed to be spherically symmetric so that a general metric of the form

$$\begin{aligned} d\sigma^2 &= e^{2\lambda} d\eta^2 - e^{2\mu} d\xi^2 - r^2 d\Omega^2 \\ d\Omega^2 &= d\theta^2 + \sin^2\theta d\phi^2 \end{aligned} \tag{1.1}$$

may be employed. The functions λ, μ, r may depend both on the time coordinate η and on the spatial coordinate ξ, but are always independent of θ and ϕ. Dimensions have been chosen so that the speed of light and the Newtonian constant of gravitation have the value unity.

From equation (1.1) and with the coordinate identifications $x^1 \equiv \xi$, $x^2 \equiv \theta$, $x^3 \equiv \phi$, and $x^4 \equiv \eta$ the components of the covariant symmetric Einstein tensor G_{ab} $(a, b = 1,2,3,4)$ may be found. It is well known that the only non-vanishing components for the above system are

$$G_{11}, G_{22} = G_{33}, G_{44}, G_{14} = G_{41} \tag{1.2}$$

The Einstein field equations are

$$G_b{}^a = -8\pi T_b{}^a \tag{1.3}$$

where $T_b{}^a$ is the energy-momentum tensor.

It will be assumed that the distribution of material is a perfect fluid so that the energy tensor takes the form

$$T_b{}^a = (\rho + p) u^a u_b - \delta_b{}^a p \tag{1.4}$$

where ρ and p are the density and pressure, respectively, and the velocity four-vector u^a $(a = 1,2,3,4)$ satisfies

$$u^a u_a = 1 \tag{1.5}$$

Since $G_{12}, G_{23}, G_{24}, G_{13}$ all vanish identically, it follows that

$$u^2 = 0, \qquad u^3 = 0 \tag{1.6}$$

so that the motion of the fluid is radial, and the components of the energy tensor that do not vanish identically are

$$\begin{aligned} T_1{}^1 &= (\rho + p)u^1 u_1 - p \\ T_2{}^2 &= T_3{}^3 = -p \\ T_4{}^4 &= (\rho + p)u^4 u_4 - p \\ T_4{}^1 &= -e^{2(\lambda - \mu)} T_1{}^4 = (\rho + p)u^1 u_4 \end{aligned} \tag{1.7}$$

2. *The Isotropy Condition*

From the simultaneous set of equations (1.7), ρ, p, u^1, and u^4 may be eliminated with the result that

$$e^{2\lambda}(T_1{}^4)^2 + e^{2\mu}(T_2{}^2 - T_1{}^1)(T_2{}^2 - T_4{}^4) = 0 \tag{2.1}$$

This equation was first obtained by Walker (1935) and arises purely from the assumptions that (i), the material is spherically symmetric and (ii), the material is a perfect fluid. Hence equation (2.1) expresses the necessary and sufficient condition that the metric (1.1) shall correspond to an isotropic system. It is for this reason that (2.1) will henceforth be called the isotropy condition.

It can easily be shown that this equation remains invariant under coordinate transformations of the type

$$\Sigma = \Sigma(\xi, \eta), \qquad \Pi = \Pi(\xi, \eta) \tag{2.2}$$

where the new radial variable Σ and the time variable Π are independent of θ and ϕ.

With Einstein's equations, (2.1) becomes

$$e^{2\lambda}(G_1{}^4)^2 + e^{2\mu}(G_2{}^2 - G_1{}^1)(G_2{}^2 - G_4{}^4) = 0 \tag{2.3}$$

and this in turn can be converted into a single nonlinear differential equation in terms of λ, μ, r and their first and second partial derivatives. One particular solution of (2.3) is given by

$$G_1{}^4 = 0 = G_4{}^1, \qquad G_1{}^1 = G_2{}^2 \tag{2.4}$$

and this, by (1.3) and (1.7), leads to

$$u^1 = 0 \tag{2.5}$$

Thus in general the coordinate system is comoving. A noncomoving system occurs when the isotropy condition is solved with $G_1{}^4 \neq 0, G_1{}^1 \neq G_2{}^2$.

The alternative solution of (2.3) in which $G_1{}^4 = 0$, $G_2{}^2 = G_4{}^4$ is rejected on physical grounds because these statements lead to $u^4 = 0$ and thus (1.5) becomes $u^1 u_1 = 1$ or $e^{2\mu}(u^1)^2 = -1$. This produces imaginary values of u^1.

The density, pressure, and velocity four-vector are obtainable from (1.7) in the form

$$p = -T_2{}^2, \qquad \rho = T_1{}^1 + T_4{}^4 - T_2{}^2 \tag{2.6}$$

$$e^{2\mu}(u^1)^2 = \frac{(T_2{}^2 - T_1{}^1)}{(T_1{}^1 - T_2{}^2) + (T_4{}^4 - T_2{}^2)} \tag{2.7}$$

$$e^{2\lambda}(u^4)^2 = \frac{(T_4{}^4 - T_2{}^2)}{(T_1{}^1 - T_2{}^2) + (T_4{}^4 - T_2{}^2)} \tag{2.8}$$

In comoving coordinates these expressions reduce to $p = -T_2{}^2$, $\rho = T_4{}^4$, $u^1 = 0$, and $e^{2\lambda}(u^4)^2 = 1$.

3. *Certain Methods of Solving Einstein's Equations*

Before Walker's discovery of equation (2.3), equation (2.4) had often been regarded as the isotropy condition as it was thought that the use of comoving coordinates was not restrictive. However, as has been shown in Section 2, this is only a particular solution of (2.3) and may under certain circumstances represent a restriction on the physical system. That this can be so may be indicated by analyzing a method of solution of Einstein's equations for the unknown functions λ, μ, and r occurring in (1.1). The method, which has for example been used by Bondi (1947), McVittie (1933), and Kustaanheimo and Qvist (1948), involves two basic assumptions. Firstly, some condition is imposed on the metric (1.1). For example Bondi assumes that $\lambda = 1$ whilst the remaining authors assume that isotropic coordinates exist so that

$$d\sigma^2 = e^{2\lambda}d\eta^2 - e^{2\mu}(d\xi^2 + \xi^2 d\Omega^2) \tag{3.1}$$

Conditions of this kind are by themselves not restrictive because the metrics employed could be converted into the form (1.1) by a coordinate transformation of type (2.2). However, when the second basic assumption, namely, comoving coordinates, is also introduced a restriction on the physical situation may occur. In the case of (3.1), for example, the equations (2.4) can be shown to yield the two equations

$$\dot{\mu}' - \dot{\mu}\lambda' = 0 \tag{3.2}$$

and

$$\mu'' + \lambda'' + \lambda'^2 - \mu'^2 - 2\lambda'\mu' - (\mu' + \lambda')/\xi = 0 \tag{3.3}$$

where the prime and the dot used here and elsewhere will represent partial derivatives with respect to ξ and η, respectively. These two coupled second-order partial differential equations may be solved exactly up to undetermined constants of integration for the two functions λ and μ. The constants can be found by introducing boundary conditions. Consequently isotropic coordinates plus a comoving frame of reference are sufficient to determine λ and μ and therefore density and pressure also from (2.6). Thus the two assumptions are sufficient to determine the physical structure of the configuration. This may not be realistic for a given problem as has been pointed out by Noerdlinger and Petrosian (1971).

In the case of Bondi (1947), when $\lambda = 1$ and a comoving frame of reference are used the two differential equations corresponding to (2.4) would involve

μ and r and their partial derivatives with respect to ξ and η. Again a restrictive physical structure may result.

In order to remove this type of restriction it is necessary to remove at least one of the coordinate restrictions. Hence more general solutions of Einstein's equations could be found in theory by considering either, (i) a comoving frame of reference with a metric unspecified in form, or (ii) a general frame of reference with a metric specified in form, or (iii) both an unspecified frame and metric.

The first alternative results in two differential equations for the three unknown functions λ, μ, r in (1.1) caused by $G_1{}^4 = 0$ and $G_1{}^1 = G_2{}^2$. Whitrow and Thompson (1967) have found gravitational potentials corresponding to this system but they needed to introduce additional assumptions in order to obtain analytic solutions. The third alternative insists on a degree of generality that is unnecessary. This is so since in theory a transformation of coordinates can be found that either changes the general frame into a comoving one [alternative (i)] or changes the general metric into one of specified form, for example isotropic coordinates [alternative (ii)].

It is the second alternative that we have adopted. In this case the metric can be chosen so as to simplify the isotropy condition (2.3), which will be treated in its full form. As with the comoving solutions dealt with by Whitrow and Thompson (1967) additional assumptions will be introduced in order to simplify the mathematical analysis, and these will be sufficient to determine the physical structure of the system.

The conditions (ii) and (iii) may be converted into (i), in other words, comoving coordinates are always possible. Suppose that (1.1) is the expression for the metric in terms of a noncomoving system and that

$$\Sigma = \Sigma(\xi, \eta), \qquad \Pi = \Pi(\xi, \eta) \tag{3.4}$$

are the radial and time coordinates of a comoving system. Then with the identifications $\Sigma \equiv \bar{x}^1, \Pi \equiv \bar{x}^4, \xi \equiv x^1, \eta \equiv x^4, \theta \equiv x^2 = \bar{x}^2, \phi \equiv x^3 = \bar{x}^3$, the metric (1.1) can be orthogonally transformed into

$$d\sigma^2 = \frac{e^{4\mu}\dot{\Sigma}^2 d\Pi^2}{\Pi'^2(\Sigma'^2 e^{2\lambda} - \dot{\Sigma}^2 e^{2\mu})} - \frac{e^{2(\lambda+\mu)} d\Sigma^2}{(\Sigma'^2 e^{2\lambda} - \dot{\Sigma}^2 e^{2\mu})} - r^2 d\Omega^2 \tag{3.5}$$

where the coefficients of $d\Pi^2, d\Sigma^2$, and $d\Omega^2$ must now be expressed as functions of Σ and Π. The condition expressing orthogonality is

$$e^{2\mu}\dot{\Sigma}\dot{\Pi} - e^{2\lambda}\Sigma'\Pi' = 0 \tag{3.6}$$

and moreover it must be assumed that

$$\Sigma'^2 e^{2\lambda} - \dot{\Sigma}^2 e^{2\mu} > 0 \tag{3.7}$$

in (3.5), so as not to violate the condition that Σ is spacelike and Π is timelike. In the comoving Π, Σ system the velocity four-vector is $\bar{u}^\alpha$ and only $\bar{u}^4$ is nonzero. But $\bar{u}^1$ is related to u^4, u^1 of the noncomoving system by

$$\bar{u}^1 = 0 = \Sigma' u^1 + \dot{\Sigma} u^4 \tag{3.8}$$

The two equations (3.6) and (3.8) determine the transformation functions Σ and Π that transform the noncomoving system into its comoving counterpart. An alternative to (3.8) is obtained by multiplying it by $8\pi(\rho + p)u_1$ and using (1.7) and (1.3) to give

$$\dot{\Sigma} G_1{}^4 - \Sigma'(G_2{}^2 - G_1{}^1) = 0 \tag{3.9}$$

Previous investigations in which the isotropy condition is treated in its full form apparently begin with the solutions obtained by Narlikar and Moghe (1935a, b) using an isotropic coordinate system, but they were not analyzed. Moghe and Sastry (1936) used noncomoving frames to consider "What happens to the Schwarzschild interior solution when it becomes nonstatic owing to some instability." The result of their investigation is a solution that is apparently only expressible in terms of a very complicated series. For many years after 1936 no published literature was identified until the recent work of Vaidya (1968). In this case the metric is written down in terms of curvature coordinates [$r = \xi$ in (1.1)]. The main characteristic of this class of solution is that the mass of the configuration is equal to $4\pi/3$ times the density multiplied by the radius cubed. Moreover, it is found that a subclass contains the noncomoving equivalent of the comoving Robertson-Walker metrics for a homogeneous universe.

4. *The Isotropy Condition for a Metric of Specified Form*

To obtain solutions of the isotropy condition it will be assumed here that the spherically symmetric metric takes the form

$$d\sigma^2 = e^{2\lambda} d\eta^2 - e^{2\mu}(d\xi^2 + f^2 d\Omega^2) \tag{4.1}$$

where f is a function of ξ alone and λ, μ are functions of both ξ and η.

The nonvanishing components of the Einstein tensor for this metric can be shown to be

$$G_1{}^1 = -e^{-2\lambda}\{2\ddot{\mu} + 3\dot{\mu}^2 - 2\dot{\lambda}\dot{\mu}\}$$

$$+ e^{-2\mu}\left\{\mu'^2 + 2\lambda'\mu' + \frac{2f'}{f}(\mu' + \lambda') + \frac{f'^2}{f^2} - \frac{1}{f^2}\right\} + \Lambda \tag{4.2}$$

$$G_2{}^2 = -e^{2\lambda}\{2\ddot{\mu} + 3\dot{\mu}^2 - 2\dot{\lambda}\dot{\mu}\}$$

$$+ e^{-2\mu}\left\{\mu'' + \lambda'' + \lambda'^2 + \frac{f'}{f}(\mu' + \lambda') + \frac{f''}{f}\right\} + \Lambda \tag{4.3}$$

$$G_4{}^4 = -3e^{-2\lambda}\dot{\mu}^2$$

$$+ e^{-2\mu}\left\{2\mu'' + \mu'^2 + \frac{4f'\mu'}{f} + \frac{2f''}{f} + \frac{f'^2}{f^2} - \frac{1}{f^2}\right\} + \Lambda \tag{4.4}$$

$$e^{2\lambda}G_1{}^4 = -e^{2\mu}G_4{}^1 = 2\{\dot{\mu}' - \lambda'\dot{\mu}\} \tag{4.5}$$

where $\dot{x} = \partial x/\partial\eta, x' = \partial x/\partial\xi$, and Λ is the cosmical constant.

From (4.2)-(4.5) it is convenient to form the following quantities:

$$G_2{}^2 - G_1{}^1 = e^{-2\mu}K_1 \tag{4.6}$$

$$G_2{}^2 - G_4{}^4 = -2e^{-2\lambda}K_4 - e^{-2\mu}K_2 \tag{4.7}$$

$$e^{2\lambda}G_1{}^4 = -e^{2\mu}G_4{}^1 = 2K_3 \tag{4.8}$$

where

$$K_1 = \mu'' + \lambda'' + \lambda'^2 - \mu'^2 - 2\lambda'\mu'$$

$$-\frac{f'}{f}(\mu' + \lambda') + \frac{f''}{f^2} - \frac{f'^2}{f^2} + \frac{1}{f^2} \tag{4.9}$$

$$K_2 = \mu'' - \lambda'' + \mu'^2 - \lambda'^2 + \frac{f'}{f}(3\mu' - \lambda')$$

$$+\frac{f''}{f} + \frac{f'^2}{f^2} - \frac{1}{f^2} \tag{4.10}$$

$$K_3 = \dot{\mu}' - \lambda'\dot{\mu}, \qquad K_4 = \ddot{\mu} - \dot{\lambda}\dot{\mu} \tag{4.11}$$

Thus from (4.6)-(4.8) the isotropy condition (2.4) may be written as

$$e^{2(\mu - \lambda)}\{4K_3{}^2 - 2K_1K_4\} - K_1K_2 = 0 \tag{4.12}$$

In addition from Einstein's equations and (4.6)-(4.8) the density, pressure, and nonvanishing components of the velocity four-vector given by (2.6), (2.10), and (2.11) become

$$8\pi p = G_2{}^2, \qquad 8\pi\rho = e^{-2\mu}K_1 - G_4{}^4$$

$$e^{2\mu}(u')^2 = \frac{e^{-2\mu}K_1{}^2}{4e^{-2\lambda}K_3{}^2 - e^{-2\mu}K_1{}^2} \tag{4.13}$$

$$e^{2\lambda}(u^4)^2 = \frac{4e^{-2\lambda}K_3{}^2}{4e^{-2\lambda}K_3{}^2 - e^{-2\mu}K_1{}^2}$$

The noncomoving condition, from (4.6) and (4.8), is satisfied when $K_1 \neq 0$ and $K_3 \neq 0$ for all values of ξ and η.

5. *Method of Solution of the Isotropy Condition*

Since the isotropy condition (4.12) is a second-order, nonlinear, partial differential equation of two variables ξ and η and three unknown functions

λ, μ, f, it is necessary to introduce a number of assumptions, so that it may be solved uniquely up to constants of integration. As has been stated in Section 1, this will be achieved by introducing purely mathematical assumptions. In presenting solutions of the equation, many different constants of integration will arise. The letters n, N, A, B (with or without numerical subscripts) are reserved for such constants. Any other constants will be denoted by the letters a, b (with or without numerical subscripts) $\xi_0 > 0, \eta_0 > 0$, and k. Whilst a, b, ξ_0, η_0 are arbitrary constants, k has special values given by 1,0, −1. On certain occasions the same letter may be used in different solutions. It is taken for granted that this letter may assume a different significance in each solution. Finally, whenever a letter is used as a subscript it will (unless otherwise stated) refer to a total derivative with respect to that subscript, thus $\beta_z = d\beta/dz$.

The first mathematical assumption concerns the nature of the line element (4.1). It will be supposed that the functions λ and μ have the form

$$\lambda = \alpha + \Psi, \qquad \mu = \beta + \psi \tag{5.1}$$

where α, β are functions of a single variable z which is dependent on both ξ and η. In addition Ψ and ψ are functions of η alone. The metric (4.1) thus becomes

$$d\sigma^2 = e^{2(\alpha + \Psi)}d\eta^2 - e^{2(\beta + \psi)}(d\xi^2 + f^2 d\Omega^2) \tag{5.2}$$

If now (5.1) is substituted into expressions (4.11) then K_3 and K_4 become

$$K_3 = \dot{z}z'(\beta_{zz} - \alpha_z\beta_z) + \beta_z\dot{z}' - \alpha_z z'\dot{\psi} \tag{5.3}$$

and

$$K_4 = \dot{z}^2(\beta_{zz} - \alpha_z\beta_z) + \beta_z(\ddot{z} - \dot{\Psi}\dot{z}) + \ddot{\psi} - \alpha_z\dot{z}\dot{\psi} - \dot{\psi}\dot{\Psi} \tag{5.4}$$

Inspection of (5.3) and (5.4) shows that they will simplify if $\dot{\psi} = a\dot{z}$ is imposed. Since ψ is a function of η alone, then by this restriction so too is $\dot{z}$. Thus $\dot{z}' = 0$ and

$$z(\xi, \eta) = h(\xi) + g(\eta) \tag{5.5}$$

where h is an arbitrary function of ξ and g is an arbitrary function of η. Therefore $\dot{\psi} = a\dot{g}$ and upon integration

$$\psi = ag \tag{5.6}$$

since the constant of integration may be incorporated in the arbitrary function $g(\eta)$. Consequently with these conditions and equations (5.3), (5.4), (4.9), (4.10) the expressions for K_1 to K_4 now become

$$\begin{aligned} K_1 &= h'^2(\alpha_{zz} + \beta_{zz} + \alpha_z^2 - \beta_z^2 - 2\alpha_z\beta_z) \\ &\quad + \left(h'' - \frac{f'h'}{f}\right)(\alpha_z + \beta_z) + \frac{f''}{f} - \frac{f'^2}{f^2} + \frac{1}{f^2} \end{aligned} \tag{5.7}$$

$$K_2 = h'^2(\beta_{zz} - \alpha_{zz} + \beta_z^2 - \alpha_z^2) + h''(\beta_z - \alpha_z) + \frac{f'h'}{f}(3\beta_z - \alpha_z) + \frac{f''}{f} + \frac{f'^2}{f^2} - \frac{1}{f^2} \tag{5.8}$$

$$K_3 = h'\dot{g}(\beta_{zz} - \alpha_z\beta_z - a\alpha_z) \tag{5.9}$$

$$K_4 = \dot{g}^2(\beta_{zz} - \alpha_z\beta_z - a\alpha_z) + (\beta_z + a)(\ddot{g} - \dot{g}\dot{\Psi}) \tag{5.10}$$

Inspection of these four relations shows that they will simplify if the following equations hold. Firstly,

$$\frac{f''}{f} - \frac{f'^2}{f^2} + \frac{1}{f^2} = 0 \tag{5.11}$$

secondly,

$$h'' - \frac{f'h'}{f} = bh'^2 \tag{5.12}$$

and lastly

$$h'' = b_1h'^2 + b_2 \tag{5.13}$$

With these restrictions, and the introduction of K_5, which is a function of z, through

$$K_5 = \beta_{zz} - \alpha_{zz} + \beta_z^2 - \alpha_z^2 - b\beta_z \tag{5.14}$$

equations (5.7) and (5.8) become

$$K_1 = h'^2\{\alpha_{zz} + \beta_{zz} + \alpha_z^2 - \beta_z^2 - 2\alpha_z\beta_z + b(\alpha_z + \beta_z)\} \tag{5.15}$$

and

$$K_2 = h'^2\{K_5 + (2b_1 - b)(2\beta_z - \alpha_z)\} + 2\{b_2(2\beta_z - \alpha_z) + f''/f\} \tag{5.16}$$

Therefore if equations (5.1), (5.6), (5.9), (5.10), and (5.14)-(5.16) are substituted into (4.12) it is found that the isotropy condition takes the form

$$\begin{aligned} &2h'^2\dot{g}^2e^{2(\beta - \alpha + ag - \Psi)}(\beta_{zz} - \alpha_z\beta_z - a\alpha_z)\{K_5 - (2a + b)\alpha_z\} \\ &- K_1\{2e^{2(\beta - \alpha + ag - \psi)}(\beta_z + a)(\ddot{g} - \dot{g}\dot{\psi}) + 2[b_2(2\beta_z - \alpha_z) + f''/f] \\ &+ h'^2[K_5 + (2b_1 - b)(2\beta_z - \alpha_z)]\} = 0 \end{aligned} \tag{5.17}$$

It is this form of the isotropy condition that will provide solutions later.

Since (5.17) is to be solved subject to the noncomoving coordinate condition, it follows that K_1 and K_3 must in general be nonvanishing. A helpful relation for showing this may be obtained from equations (5.9), (5.14), and (5.15), namely,

$$2h'K_3 - \dot{g}K_1 = \dot{g}h'^2\{K_5 - (2a + b)\alpha_z\} \tag{5.18}$$

Many of the subsequent solutions satisfy the condition $K_5 - (2a + b)\alpha_z = 0$; for these it is only necessary to calculate one of K_1 and K_3 to verify the non-comoving character of a solution.

In order to solve equations (5.11)-(5.13) simultaneously it is convenient to define a function $\mathscr{S}_k(x)$ given by

$$\begin{aligned} \mathscr{S}_k(x) &= \sin x, & k &= +1 \\ &= x, & k &= 0 \\ &= \sinh x, & k &= -1 \end{aligned} \tag{5.19}$$

If, further, $\mathscr{C}_k(x)$ is defined by

$$\mathscr{S}_k(x) = \frac{d\mathscr{C}_k(x)}{dx} \tag{5.20}$$

and it is easily proved that

$$\begin{aligned} &\frac{d^2\mathscr{S}_k(x)}{dx^2} = \frac{d\mathscr{C}_k(x)}{dx} = -k\,\mathscr{S}_k(x) \\ &\mathscr{C}_k^2(x) + k\mathscr{S}_k^2(x) = 1 \\ &\mathscr{C}_k(2x) = 2\mathscr{C}_k^2(x) - k \\ &2\,\mathscr{S}_k(x)\mathscr{C}_k(x) = \mathscr{S}_k(2x) \end{aligned} \tag{5.21}$$

Equation (5.11) has as its first integral

$$f'^2 = 1 - kn_1^2 f^2$$

where $n_1 \neq 0$. The three values of k mean that there are three types of solution for f, which can be written compactly in the form

$$f = (1/n_1)\,\mathscr{S}_k(n_1\xi + n_2)$$

When the metric (5.2) is inspected, the requirements of the geometry of a sphere show that $f = 0$ at $\xi = 0$. This means that $n_2 = 0$ and so

$$f = (1/n_1)\,\mathscr{S}_k(n_1\xi) \tag{5.22}$$

The expression for h will now be determined by considering two cases defined by $b_1 = 0$ and $b_1 \neq 0$.

Case I. When $b_1 = 0$ the solution of (5.13) is

$$h = b_2\xi^2 + N_1\xi + N_2 \tag{5.23}$$

Therefore when (5.22) and (5.23) are substituted in (5.12) it follows that

$$b_2 - n_1(b_2\xi + N_1)\,\frac{\mathscr{C}_k(n_1\xi)}{\mathscr{S}_k(n_1\xi)} \equiv b(b_2\xi + N_1)^2$$

which must be true for all ξ. Thus the only possibility is $k = 0$, $b = 0$, $N_1 = 0$. Hence the complete solution is

$$\begin{aligned} &f = \xi, \qquad h = b_2\xi^2/2 + N_2 \\ &k = 0, \qquad b = 0, \qquad b_1 = 0 \end{aligned} \tag{5.24}$$

Case II. Since now $b_1 \neq 0$, the substitution

$$h = -(1/b_1)\ln H \tag{5.25}$$

where H is a function of ξ, reduces (5.13) to

$$H'' + b_1 b_2 H = 0$$

This leads to the solution

$$H = N_3\, \mathscr{S}_k(b_3 x) \tag{5.26}$$

where

$$x = \xi + N_2 \tag{5.27}$$

and

$$\begin{aligned} &kb_3^2 = b_1 b_2, \qquad k = 1, -1 \quad \text{when } b_2 \neq 0 \\ &b_3 = 1, \qquad\quad\;\; k = 0 \qquad\;\; \text{when } b_2 = 0 \end{aligned} \tag{5.28}$$

Equations (5.25) and (5.26) together lead to

$$h = -(1/b_1)\ln\{N_3\, \mathscr{S}_k(b_3 x)\} \tag{5.29}$$

In order to show that (5.29) satisfies (5.12), two subcases will be considered and these are defined by $k = 0$ and $k = \pm 1$.

Subcase II(i). When $k = 0$, then $b_2 = 0$, $b_3 = 1$ by (5.28) and

$$f = \xi, \qquad h = -(1/b_1)\ln(N_3 x) \tag{5.30}$$

Also with the aid of (5.30), equation (5.12) becomes the identity

$$\frac{1}{\xi} = \frac{b - b_1}{b_1}\,\frac{1}{\xi + N_2} \tag{5.31}$$

which is satisfied only when $N_2 = 0$ and $b = 2b_1 \neq 0$. Therefore, the complete solution for f and h in this subcase is

$$\begin{aligned} &f = \xi, \qquad h = -(1/b_1)\ln(N_3\xi) \\ &b = 2b_1 \neq 0, \qquad b_2 = 0, \qquad k = 0 \end{aligned} \tag{5.32}$$

Subcase II(ii). This subcase is defined by the first of alternatives (5.28) so that $b_2 \neq 0$, $kb_3^2 = b_1 b_2$, $k = \pm 1$. Thus we have

$$f = \frac{1}{n_1}\mathscr{S}_k(n_1\xi), \qquad h = -\frac{1}{b_1}\ln\{N_3\, \mathscr{S}_k(b_3 x)\} \tag{5.33}$$

where $x = \xi + N_2$ and so (5.12) becomes the identity

$$n_1 \frac{\mathscr{C}_k(n_1\xi)}{\mathscr{S}_k(n_1\xi)} \equiv \frac{b_3}{b_1} \frac{\{b\mathscr{C}_k^2(b_3x) - 1\}}{\mathscr{C}_k(b_3x)\,\mathscr{S}_k(b_3x)}$$

or with the aid of (5.21)

$$n_1 \frac{\mathscr{C}_k(n_1\xi)}{\mathscr{S}_k(n_1\xi)} \equiv \frac{b_3}{b_1} \frac{b\mathscr{C}_k(2b_3x) + b - 2b_1}{\mathscr{S}_k(2b_3x)}$$

This identity is satisfied by $b = 2b_1, N_2 = 0, n_1 = 2b_3$, and so for this case the full solution for f and h is

$$f = \frac{1}{n_1}\mathscr{S}_k(n_1\xi), \qquad h = -\frac{1}{b_1}\ln\left\{N_3\,\mathscr{S}_k\left(\frac{n_1\xi}{2}\right)\right\}$$
$$b = 2b_1 \neq 0, \qquad kn_1^2 = 4b_1b_2, \qquad k = \pm 1 \tag{5.34}$$

The functions f and h have thus been found and it is then possible to proceed to the solutions of the isotropy condition (5.17).

6. *Solutions A*

In this section all solutions presented are defined by the conditions of case I in Section 5 so that equation (5.24) holds. Thus the metric (5.2) with (5.6) and (5.24) becomes

$$d\sigma^2 = e^{2(\alpha + \Psi)}d\eta^2 - e^{2(\beta + ag)}(d\xi^2 + \xi^2 d\Omega^2) \tag{6.1}$$

and the isotropy condition (5.17) is now

$$2h'^2\dot{g}^2 e^{2(\beta - \alpha + ag - \Psi)}(\beta_{zz} - \alpha_z\beta_z - a\alpha_z)\,\{K_5 - 2a\alpha_z\}$$
$$-K_1\{2e^{2(\beta - \alpha + ag - \Psi)}(\beta_z + a)\,(\ddot{g} - \dot{g}\dot{\Psi}) + 2b_2(2\beta_z - \alpha_z) + h'^2K_5\} = 0 \tag{6.2}$$

where K_1, K_5, given by (5.15) and (5.14), are

$$K_1 = h'^2\{\alpha_{zz} + \beta_{zz} + \alpha_z^2 - \beta_z^2 - 2\alpha_z\beta_z\} \tag{6.3}$$

$$K_5 = \beta_{zz} - \alpha_{zz} + \beta_z^2 - \alpha_z^2 \tag{6.4}$$

The expression for z is given by (5.5) with (5.24), and so

$$z = b_2\xi^2/2 + N_2 + g(\eta) \tag{6.5}$$

The foregoing expressions lead to the following subcases.

Class A(i). A class of solutions may be defined through the assumptions

$$a = 0, \qquad \alpha = \beta, \qquad \beta_{zz} - \beta_z^2 \neq 0 \tag{6.6}$$

Thus from (6.3) and (6.4)

$$K_1 = 2h'^2(\beta_{zz} - \beta_z^2), \qquad K_5 = 0 \tag{6.7}$$

and so the isotropy condition (6.2) is satisfied by

$$e^{-2\Psi}(\ddot{g} - \dot{g}\dot{\Psi}) + b_2 = 0 \tag{6.8}$$

The corresponding line element (6.1) is

$$d\sigma^2 = e^{2\beta}\{e^{2\Psi}d\eta^2 - (d\xi^2 + \xi^2 d\Omega^2)\} \tag{6.9}$$

If a new time variable $\bar{\eta}$ is introduced through

$$\bar{\eta} = \int e^{\Psi(\eta)}d\eta \tag{6.10}$$

the equation (6.8) becomes

$$\frac{d^2 g}{d\bar{\eta}^2} + b_2 = 0$$

whence, without loss in generality,

$$g = -b_2\bar{\eta}^2/2 - N_2 \tag{6.11}$$

Hence from (6.5) with (6.10) and (6.11) the expression for z is

$$z = b_2(\bar{\xi}^2 - \bar{\eta}^2)/2$$

Thus if a new constant is defined by

$$\epsilon/\xi_0^2 = b_2/2, \qquad \epsilon = \pm 1$$

and the bars are omitted, the metric (6.9) with (6.10) is conformally Minkowskian so that

$$d\sigma^2 = e^{2\beta}(d\eta^2 - d\xi^2 - \xi^2 d\Omega^2) \tag{6.12}$$

$$z = \epsilon(\xi^2 - \eta^2)/\xi_0^2 \tag{6.13}$$

and β is an undetermined function of z.

Solutions of this kind imply that the coordinate system is noncomoving, since by equations (5.18), (5.24), (6.6), and (6.7)

$$2\xi K_3 = -\eta K_1 = -8\eta\xi^2(\beta_{zz} - \beta_z^2)/\xi_0^2 \tag{6.14}$$

and thus $K_1 \neq 0, K_3 \neq 0$.

Class A(ii). This solution follows from the four independent assumptions

$$\ddot{g} - \dot{g}\dot{\Psi} = 0, \qquad 2\beta_z - \alpha_z = 0, \qquad K_5 = 0, \qquad a = 0 \tag{6.15}$$

which clearly satisfy (6.2). The first equation leads to

$$e^{\Psi} = n_3\dot{g} \tag{6.16}$$

while the second and third with (6.4) yield

$$e^{\beta} = B(z + A)^{1/3} \tag{6.17}$$

$$e^{\alpha} = B_1^2(z+A)^{2/3} \tag{6.18}$$

where A, B, B_1 are the constants of integration. Thus with equations (6.15)-(6.18) the metric (6.1) becomes

$$d\sigma^2 = (z+A)^{4/3}(n_3B_1^2dg)^2 - (z+A)^{2/3}B^2(d\xi^2 + \xi^2 d\Omega^2) \tag{6.19}$$

This expression may be simplified by the introduction of a new time variable $\bar{\eta}$ and a radial coordinate $\bar{\xi}$ defined by

$$n_3B_1^2dg = \epsilon d\bar{\eta}, \qquad B\xi = \bar{\xi} \tag{6.20}$$

where $\epsilon = \pm 1$. Integration of the first of these equations with a particular choice of constant of integration yields

$$g = \epsilon\bar{\eta}/n_3B_1^2 - N_2 - A$$

and so with (6.5) and (6.20) the expression for $\bar{z} \equiv z + A$ is now

$$z \equiv z + A = \frac{b_2\bar{\xi}^2}{2B^2} + \frac{\epsilon\bar{\eta}}{n_3B_1^2}$$

Since b_2 may be positive or negative, it is possible to define two new constants $\bar{\xi}_0, \bar{\eta}_0$ by

$$\frac{\epsilon_1}{\bar{\xi}_0^2} = \frac{b_2}{2B^2}, \qquad \epsilon_1 = \pm 1, \qquad \bar{\eta}_0 = n_3B_1^2 > 0$$

Then the bars may be omitted and the metric written as

$$d\sigma^2 = z^{4/3}d\eta^2 - z^{2/3}(d\xi^2 + \xi^2 d\Omega^2) \tag{6.21}$$

$$z = \epsilon_1\xi^2/\xi_0^2 - \epsilon\eta/\eta_0 \tag{6.22}$$

This solution may be shown to be noncomoving from equations (4.1), (4.11), (4.9), (6.21), (6.22) with the result that

$$K_1 = \frac{4\epsilon_1\eta_0}{\epsilon\xi_0^2} \quad \xi K_3 = -\frac{40\xi^2}{9\xi_0^4 z^2} \tag{6.23}$$

and so in general K_1 and K_3 are nonzero.

7. *Solutions B*

The defining equations for these solutions are given by (5.32) with the additional assumption that $b_1 = -1$. Moreover, since it is always possible to redefine the radial coordinate ξ by $\bar{\xi} = N_3\xi$, it is not restrictive to take $N_3 = 1$ and so the equations (5.32) are now

$$\begin{aligned} &f = \xi, \qquad &&h = \ln\xi, \qquad &&k = 0 \\ &b = -2, \qquad &&b_1 = -1, \qquad &&b_2 = 0 \end{aligned} \tag{7.1}$$

and the metric (5.2) with (5.6) and (7.1) will now be written

$$d\sigma^2 = e^{2[\alpha + \Psi(\bar{\eta})]} d\bar{\eta}^2 - e^{2\beta + 2ag(\bar{\eta})}(d\xi^2 + \xi^2 d\Omega^2) \tag{7.2}$$

and hence with (7.1) and $\dot{g} = dg/d\eta$ the isotropy condition (5.17) becomes

$$\begin{aligned} &2\xi^2 \dot{g}^2 e^{2(\beta - \alpha + ag - \Psi)}(\beta_{zz} - \alpha_z\beta_z - a\alpha_z)\{K_5 - 2(a-1)\alpha_z\} \\ &\quad - (\xi^2 K_1)\{2\xi^2 e^{2(\beta - \alpha + ag - \Psi)}(\beta_z + a)(\ddot{g} - \dot{g}\dot{\Psi}) + K_5\} = 0 \end{aligned} \tag{7.3}$$

The expressions for K_1, K_5 given by (5.15) and (5.14) are now

$$\xi^2 K_1 = \alpha_{zz} + \beta_{zz} + \alpha_z^2 - \beta_z^2 - 2\alpha_z\beta_z - 2(\alpha_z + \beta_z) \tag{7.4}$$

$$K_5 = \beta_{zz} - \alpha_{zz} + \beta_z^2 - \alpha_z^2 + 2\beta_z \tag{7.5}$$

while from (5.5) and (7.1) the expression for z is

$$e^z = \xi e^{g(\bar{\eta})} \tag{7.6}$$

A further condition relating the functions $g(\bar{\eta})$ and $\Psi(\bar{\eta})$ will now be imposed, namely,

$$\dot{g}^2 e^{2(ag - \Psi)} = e^{2g}$$

Therefore,

$$\left.\begin{aligned} &\dot{g}^2 e^{2g(a-1)} = e^{2\Psi} \\ &\text{and} \\ &(\ddot{g} - \dot{g}\dot{\Psi})e^{2(ag - \Psi)} = -(a-1)e^{2g} \end{aligned}\right\} \tag{7.7}$$

Hence, upon multiplication by ξ^2 it is found that

$$\left.\begin{aligned} \xi^2 \dot{g}^2 e^{2(ag - \Psi)} &= e^{2z} \\ \xi^2(\ddot{g} - \dot{g}\dot{\Psi})e^{2(ag - \Psi)} &= -(a-1)e^{2z} \end{aligned}\right\} \tag{7.8}$$

This means that the isotropy condition (7.3) with (7.4) and (7.5) now becomes

$$\begin{aligned} &2e^{2(\beta - \alpha + z)}(\beta_{zz} - \alpha_z\beta_z - a\alpha_z)\{\beta_{zz} - \alpha_{zz} + \beta_z^2 \\ &\quad - \alpha_z^2 + 2\beta_z - 2(a-1)\alpha_z\} - \{\alpha_{zz} + \beta_{zz} + \alpha_z^2 - \beta_z^2 \\ &\quad - 2\alpha_z\beta_z - 2(\alpha_z + \beta_z)\}\{\beta_{zz} - \alpha_{zz} + \beta_z^2 \\ &\quad - \alpha_z^2 + 2\beta_z - 2(a-1)(\beta_z + a)e^{2(\beta - \alpha + z)}\} = 0 \end{aligned} \tag{7.9}$$

This equation therefore involves only the combination z, of $\bar{\xi}$ and $\bar{\eta}$. It is in fact a generalization of equation (4) of Narlikar and Moghe (1935b).

The line element (7.2) with (7.7) is now

$$d\sigma^2 = e^{2\alpha + 2g(a-1)}(\epsilon dg)^2 - e^{2\beta + 2ag}(d\xi^2 + \xi^2 d\Omega^2) \tag{7.10}$$

where $\epsilon = 1, -1$. New variables (η, ω) are defined by

$$\epsilon g(\bar{\eta}) = \eta, \qquad \omega = \ln\xi \tag{7.11}$$

Hence (7.10) may be written in the form

$$d\sigma^2 = e^{2(a-1)\epsilon\eta}\{e^{2\alpha}d\eta^2 - e^{2(\beta+z)}(d\omega^2 + d\Omega^2)\} \tag{7.12}$$

$$z = \ln\xi e^{g(\bar{\eta})} = \ln\xi + \epsilon\eta = \omega + \epsilon\eta \tag{7.13}$$

The form of the metric (7.12), which is also employed by Taub (1968), is best regarded as a mathematical device, since from the physical point of view the variable ω does not present the normal properties of a radial coordinate. For example, the center of the distribution $\xi = 0$ does not correspond to $\omega = 0$ but rather to $\omega = -\infty$. Therefore, in any analysis of solutions of (7.9) the (ξ, η) system should be ultimately regarded as the physically significant system.

Inspection of the equation (7.9) shows that its left-hand side consists of the product of two factors when $a = 1$. Either of the factors may be equated to zero, and both possibilities have been found to produce solutions. However, the complete analysis is lengthy and, for the sake of brevity, these solutions are omitted and will be treated in a later paper.

A simpler class of solutions of (7.9) arises from the assumption

$$\beta - \alpha + z = a_0 \tag{7.14}$$

where a_0 is a constant. The metric (7.12) becomes

$$d\sigma^2 = e^{2(\beta+z)+2\epsilon(a-1)\eta}(e^{-2a_0}d\eta^2 - d\omega^2 - d\Omega^2) \tag{7.15}$$

while (7.5) is

$$K_5 = -1 \tag{7.16}$$

The isotropy condition (7.9) is

$$\begin{aligned} &2\{1 + e^{2a_0}(1 - 4a + 2a^2)\}\beta_{zz} + 2(e^{2a_0} - 1)\beta_z^2 \\ &+ 4(ae^{2a_0} - 1)\beta_z + 2a^2e^{2a_0} - 1 = 0 \end{aligned} \tag{7.17}$$

It is evident that this equation cannot be satisfied if $a = 1$ and also $a_0 = 0$. But the equation may be readily solved in terms of elementary functions in the three subcases

(i) $a_0 = 0$
(ii) $a_0 \neq 0$ and $1 + e^{2a_0}(1 - 4a + 2a^2) \neq 0$
(iii) $a_0 \neq 0$ and $1 + e^{2a_0}(1 - 4a + 2a^2) = 0$

As an illustration, the details for the $a_0 = 0$ case are given.

The solution of (7.17) is

$$\begin{aligned} 2(\beta + z) &= Ae^{-n_0/L}e^{(z+n_0)/L} + B(z + n_0)/L \\ z + n_0 &= \omega + \epsilon\eta + n_0 \end{aligned} \tag{7.18}$$

where A, n_0 are the constants of integration and B, L, and an additional constant C, depend on a and are defined by

$$L = 1 - a, \qquad B = \tfrac{1}{2}(1 + 2L^2), \qquad C = \tfrac{1}{2}(1 - 2L^2) \tag{7.19}$$

Clearly, n_0 can be abolished by a suitable adjustment of the origin of η time and hence can be equated to zero in (7.18). Thus the metric (7.15) is

$$d\sigma^2 = \exp(Ae^{z/L} + Bz/L - 2L\epsilon\eta)\,(d\eta^2 - d\omega^2 - d\Omega^2)$$
$$z = \omega + \epsilon\eta \tag{7.20}$$

From equations (7.4), (5.9), (7.1), (7.11), and (7.18) it is found that

$$\xi^2 K_1 = -(1/2L^2)\{Ae^{z/L} - C\}^2 \tag{7.21}$$

$$\xi K_3 = -(\epsilon/4L^2)\{Ae^{z/L} - C\}\{Ae^{z/L} - B\} \tag{7.22}$$

which verify that the solution is noncomoving.

8. *Solutions C*

The defining equations for this solution are given by (5.34), with the additional assumption that $b_1 = -1$. Therefore, equations (5.34) with the expression for e^z are now

$$f = (1/n_1)\,\mathscr{S}_k(n_1\xi), \qquad h = \ln\{N_3\,\mathscr{S}_k(n_1\xi/2)\}$$
$$b = -2, \qquad b_1 = -1, \qquad k = \pm 1, \qquad kn_1{}^2 = -4b_2 \tag{8.1}$$
$$e^z = e^{h(\xi) + g(\bar\eta)} = N_3\,\mathscr{S}_k(n_1\xi/2)e^{g(\bar\eta)}$$

where $\mathscr{S}_k(x)$ is defined by (5.19). Hence the metric (5.2), with (5.6) and (8.1), is

$$d\sigma^2 = e^{2(\alpha + \Psi)}d\bar\eta^2 - (1/n_1{}^2)e^{2[\beta + ag(\bar\eta)]}(n_1{}^2 d\xi^2 + \mathscr{S}_k^2(n_1\xi)d\Omega^2) \tag{8.2}$$

and the isotropy condition (5.17) becomes

$$\begin{aligned}&2h'^2\dot g^2 e^{2(\beta - \alpha + ag - \Psi)}(\beta_{zz} - \alpha_z\beta_z - a\alpha_z)\{K_5 - 2(a-1)\alpha_z\}\\ &\quad - K_1\{2e^{2(\beta - \alpha + ag - \Psi)}(\beta_z + a)(\ddot g - \dot g\dot\Psi) + h'^2 K_5\\ &+ 2[b_2(2\beta_z - \alpha_z) - n_1{}^2 k]\} = 0\end{aligned} \tag{8.3}$$

where $\dot g = dg/d\bar\eta$. The expressions for K_5 and K_1 from (5.14) and (5.15) are

$$K_5 = \beta_{zz} - \alpha_{zz} + \beta_z{}^2 - \alpha_z{}^2 + 2\beta_z \tag{8.4}$$

$$K_1 = h'^2\{\alpha_{zz} + \beta_{zz} + \alpha_z{}^2 - \beta_z{}^2 - 2\alpha_z\beta_z - 2(\alpha_z + \beta_z)\} \tag{8.5}$$

A solution of (8.3) can be obtained with the following additional assumptions:

$$\ddot g - \dot g\dot\Psi = 0, \qquad a = 1, K_5 = 0, \qquad b_2(2\beta_z - \alpha_z) - n_1{}^2 k = 0 \tag{8.6}$$

From the first of these we have

$$e^\Psi = n_3\dot g \tag{8.7}$$

while the last with $kn_1^2 = -4b_2$ leads to

$$2\beta_z - \alpha_z + 4 = 0 \tag{8.8}$$

so that from (8.4) the assumption $K_5 = 0$ yields

$$e^{2\beta} = n_4^2 e^{-4z}(1 + Ae^{-2z})^{2/3} \tag{8.9}$$

and so

$$e^{2\alpha} = n_5^2(1 + Ae^{-2z})^{4/3} \tag{8.10}$$

The constant A may be positive or negative. On substitution of (8.7), (8.9), and (8.10) into (8.2) it is evident that no generality is lost in writing

$$g + \ln n_4 = \epsilon\eta/\eta_0, \qquad \epsilon = \pm 1$$

$$\eta_0 = n_3 n_5, \qquad n_1 = 1, \qquad N_3 = n_4$$

and so the metric finally becomes

$$d\sigma^2 = (1 + Ae^{-2z})^{4/3} d\eta^2 - e^{2\epsilon\eta/\eta_0 - 4z}(1 + Ae^{-2z})\,[d\xi^2 + \mathscr{S}_k^2(\xi)d\Omega^2]$$
$$e^z = \mathscr{S}_k(\xi/2)e^{\epsilon\eta/\eta_0} \tag{8.11}$$

If (5.18) is now evaluated and equations (8.1), (8.5), (8.8)-(8.10) are also used, then it can be shown that

$$\begin{aligned} \dot{g}K_1 = 2h'K_3 &= -10h'^2\dot{g}(\beta_z + 2)^2 \\ &= -\frac{10\epsilon}{9\eta_0}\frac{\mathscr{C}_k^2(\xi/2)}{\mathscr{S}_k^2(\xi/2)}\frac{A^2e^{-4z}}{(1 + Ae^{-2z})^2} \end{aligned} \tag{8.12}$$

hence verifying that this solution is noncomoving.

9. *Solution by Separation of Variables*

The method of solution of the isotropy condition to be presented here is unrelated to any of the methods previously employed. It will now be assumed that the metric is

$$d\sigma^2 = e^{2(\alpha + \Psi)}d\eta^2 - e^{2(\beta + \psi)}(d\xi^2 + f^2 d\Omega^2) \tag{9.1}$$

where α, β and f are functions of ξ alone while Ψ, ψ are functions of η only. With these conditions (4.9)-(4.11) become

$$K_1 = \beta'' + \alpha'' + \alpha'^2 - \beta'^2 - 2\alpha'\beta' - \frac{f'}{f}(\alpha' + \beta') + \frac{f''}{f} - \frac{f'^2}{f^2} + \frac{1}{f^2} \tag{9.2}$$

$$K_2 = \beta'' - \alpha'' + \beta'^2 - \alpha'^2 + \frac{f'}{f}(3\beta' - \alpha') + \frac{f''}{f} - \frac{f'^2}{f^2} - \frac{1}{f^2} \tag{9.3}$$

$$K_3 = -\alpha'\dot{\psi}, \qquad K_4 = \ddot{\psi} - \dot{\psi}\dot{\Psi} \tag{9.4}$$

where $\dot{x} = dx/d\eta$, $y' = dy/d\xi$. Therefore, the isotropy condition (4.12) may be written as

$$e^{2(\beta - \alpha + \psi - \Psi)}\{4(\dot{\psi}\alpha')^2 - 2(\ddot{\psi} - \dot{\psi}\dot{\Psi})K_1\} - K_1K_2 = 0 \tag{9.5}$$

This equation will be solved by means of the assumption that

$$K_1 = 2a\alpha'^2 \tag{9.6}$$

where a is a nonzero constant. Substitution of (9.6) into (9.5) turns the isotropy condition into

$$e^{2(\psi - \Psi)}\{\dot{\psi}^2 - a\ddot{\psi} + a\dot{\psi}\dot{\Psi}\} = (aK_2/2)e^{2(\alpha - \beta)} \tag{9.7}$$

Since the left-hand side of (9.7) depends only on the variable η and the right-hand side on ξ, the equation is separable as follows:

$$\dot{\psi}^2 - a\ddot{\psi} + a\dot{\psi}\dot{\Psi} = (k/b^2)e^{2(\Psi - \psi)} \tag{9.8}$$

$$aK_2 = (2k/b^2)e^{2(\beta - \alpha)} \tag{9.9}$$

where b is an arbitrary constant, and $k = 1$ or -1.

The equation (9.9) is identically satisfied by

$$\beta = \alpha - \ln f, \qquad 2k/b^2 = -a \tag{9.10}$$

and then (9.2) and (9.6) yield

$$2\alpha'' - 2(a+1)\alpha'^2 + 2\alpha' f'/f + f^{-2} = 0 \tag{9.11}$$

Hence the condition of isotropy is satisfied, and the coefficients of the metric are determined, by any simultaneous solutions of equations (9.11) and (9.8) under the conditions (9.10). To reach these, it is convenient to replace ξ, η by ω, τ where

$$\frac{d\omega}{d\xi} = \frac{1}{f(\xi)}, \qquad \frac{d\tau}{d\eta} = \exp\{\Psi(\eta) - \psi(\eta)\} \tag{9.12}$$

and then (9.11), (9.8) become, respectively,

$$\alpha_{\omega\omega} - (\alpha + 1)\alpha_\omega{}^2 + \tfrac{1}{2} = 0 \tag{9.13}$$

$$\psi_{\tau\tau} - [(a+1)/a]\,\psi_\tau{}^2 - \tfrac{1}{2} = 0 \tag{9.14}$$

where $\alpha_\omega = d\alpha/d\omega$, $\psi_\tau = d\psi/d\tau$. The metric (9.1) is now

$$\begin{aligned} d\sigma^2 &= e^{2(\alpha + \psi)}(d\tau^2 - d\omega^2 - d\Omega^2) \\ &= e^{2(\beta + \psi)}(f^2 d\tau^2 - d\xi^2 - f^2 d\Omega^2) \end{aligned} \tag{9.15}$$

where f is an arbitrary function of ξ.

Clearly, when $a = -1$, the equations (9.13) and (9.14) yield

$$e^{\alpha + \psi} = Ae^{(\tau^2 - \omega^2)/2}, \qquad A = \text{const}$$

with suitable definitions of the origins of ω and τ. On the other hand, if $a + 1 \neq 0$, the functions α and ψ are interlocking hyperbolic or trigonometric functions of ω and τ, respectively.

Since, by (9.6) and (9.4),

$$K_1 = 2a\alpha_\omega{}^2 \left(\frac{d\omega}{d\xi}\right)^2, \qquad K_3 = -\alpha_\omega \psi_\tau \frac{d\omega}{d\xi}\frac{d\tau}{d\eta}$$

it follows that neither K_1 nor K_3 can vanish identically. Therefore, all cases considered in this section are noncomoving.

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References

Bondi, H. (1947). *Monthly Notices of the Royal Astronomical Society*, **107**, 410.

Bonnor, W. B., and Faulkes, M. C. (1967). *Monthly Notices of the Royal Astronomical Society*, **137**, 239.

Cahill, M. E., and Taub, A. H. (1971). *Communications in Mathematical Physics*, **21**, 1.

Kruskal, M. D. (1960). *Physical Review*, **119**, 1743.

Kustaanheimo, P., and Qvist, B. (1948). *Societas Scientiarum Fennica*, Commentationes Physico-Mathematicae, **XIII**, 16.

McVittie, G. C. (1933). *Monthly Notices of the Royal Astronomical Society*, **93**, 325.

McVittie, G. C. (1967). *Annales de l'Institut Henri Poincaré*, **6**, 1.

Moghe, D. N., and Sastry, R. V. (1936). *Proceedings of the National Academy of Science, India*, **95**, 91.

Narlikar, V. V., and Moghe, D. N. (1935a). *Monthly Notices of the Royal Astronomical Society*, **95**, 735.

Narlikar, V. V., and Moghe, D. N. (1935b). *Philosophical Magazine*, **20**, 7th Series, 1104.

Noerdlinger, P. D., and Petrosian, V. (1971). *Astrophysical Journal*, **168**, 1.

Taub, A. H. (1968). *Annales de l'Institut Henri Poincaré*, 9, No. 2, 153.

Vaidya, P. C. (1968). *Physical Review*, **174**, 1615.

Walker, A. G. (1935). *Quarterly Journal of Mathematics*, **6**, 81.

Whitrow, G. J., and Thompson, I. H. (1967). *Monthly Notices of the Royal Astronomical Society*, **136**, 207.